

On the relation between extremal elasticity tensors with orthotropic symmetry and extremal polynomials.

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Abstract

We prove that an elasticity tensor with orthotropic symmetry is extremal if the determinant of its acoustic tensor is an extremal polynomial that is not a perfect square.

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1 Introduction

A necessary condition for a body containing a linearly elastic homogeneous material with elasticity tensor C to be stable when the displacement is fixed at the boundary is the Legendre-Hadamard condition that the quadratic form associated with C , $f(\xi) = (C\xi; \xi)$ be rank-one convex, i.e.

$$f(x \otimes y) = \sum_{ijkl=1}^3 x_i y_j C_{ijkl} x_k y_l \geq 0 \quad \forall x, y.$$

If one has equality for some non-zero x, y then shear bands can form. Associated with f is the y -matrix (acoustic tensor), $T(y)$ with matrix elements

$$T_{ik}(y) = \sum_{j\ell=1}^3 y_j C_{ijk\ell} y_\ell \geq 0 \quad \forall x, y,$$

so rank-one convexity is equivalent to $T(y)$ being positive semi-definite for all y , which ensures real-valued wave speeds. The elasticity tensor C need not be positive semi-definite for this condition to be satisfied, i.e $f(\xi)$ need not be convex.

In this paper we show there is an interesting connection between extremal polynomials and extremal elasticity tensors which are at the boundary of being rank-one convex.

In general for quadratic functions $f(\xi) = (M\xi; \xi)$, with M not necessarily having the symmetry of elasticity tensors, Van Hove [23,24] proved that rank-one convexity is equivalent to the condition of quasiconvexity introduced by Morrey [15,16]. Due to this, and since we are only dealing with quadratic functions we will use the terms quasiconvexity and rank-one convexity interchangeably. (If the fields were not gradients but had different differential constraints then rank-one convexity is no longer appropriate but quasiconvexity is appropriate, and hence we have a preference for the term quasiconvexity). Ball introduced the condition of polyconvexity and proved it to be an intermediate condition between convexity and quasiconvexity [2]. In the quadratic case polyconvexity is equivalent [6, page 192, Lemma 5.27] to f being the sum of a convex function and a null-Lagrangian, which in the quadratic case is a function f such that $f(x \otimes y)$ vanishes for all x and y . There exist quadratic forms that are quasiconvex but not polyconvex, as shown by Terpstra in [20]. Explicit examples were given by Serre [21,22], see also Ball [3], and an especially simple example is given in [7]. A special case of quasiconvex quadratic forms are the so called extremal ones introduced by Milton in [12, page 87], see also [13, section 25.2]. This and two alternative definitions of extremals were used in [7]. In this work we will use a definition which is equivalent to the original definition:

Definition 1.1. *A quadratic quasiconvex form is called an extremal if one cannot subtract a rank-one form from it while preserving the quasiconvexity of the form.*

If a quadratic form $f(\xi) = (C\xi; \xi)$ is extremal and does not depend on the antisymmetric part of ξ we call C an extremal elasticity tensor. We prove that an elasticity tensor with orthotropic symmetry is extremal if the determinant of the y -matrix (acoustic tensor) is an extremal polynomial that is not a perfect square. The problem of characterizing all such extremals is a task for the future.

2 Motivations for studying extremals

One motivation for studying extremals comes when bounding, using the translation method, the elastic energy in a multiphase phase periodic composite with known volume fractions of the phases. Then it is always best to use translations C such that the quadratic form $f(\xi) = (C\xi; \xi)$ is an extremal [12, page 87], see also [13, section 25.2]. Extremals (with an alternative definition of extremal: one cannot subtract a symmetrized rank-one form from it while preserving the quasiconvexity) were used by Allaire and Kohn [1] in this way to bound the elastic energy of two phase composites with isotropic phases.

Extremals may also be important for obtaining sharp geometry independent estimates with Dirichlet boundary conditions of the elastic energy stored within say a two-phase body Ω (where by geometry independent we mean independent of the distribution of the phases in the body, not independent of the shape of the body). The ensuing analysis is an extension of the ideas of Tartar and Murat [18,17,19] and Lurie and Cherkaev [10, 11] for bounding the effective moduli of composite materials using the translation method and that of Kang and Milton [9] for bounding the volume fractions of materials in a two-phase body.

Let $\tilde{C}(x)$ denote the elasticity tensor taking the positive definite value C_1 in phase 1 and the positive definite value C_2 in phase 2. With Dirichlet boundary conditions $\tilde{u} = u_0$ on $\partial\Omega$, the elastic energy is

$$\tilde{W}(u_0) = \frac{1}{2} \int_{\Omega} (\tilde{C} \nabla \tilde{u}; \nabla \tilde{u}) dx \quad (1)$$

where the stress $\tilde{C} \nabla \tilde{u}$ is symmetric and only depends on the strain $\tilde{\epsilon}(x) = [\nabla \tilde{u}(x) + (\nabla \tilde{u}(x))^T]/2$, since $\tilde{C}A = 0$ when A is antisymmetric. Let the quadratic form associated with C be quasiconvex and chosen so that $\tilde{C}(x) - C$ is positive semidefinite, while $C_1 - C$ and/or $C_2 - C$ is degenerate (on the space of symmetric matrices when C has the symmetries of elasticity tensors). Writing $\tilde{C}(x) = [\tilde{C}(x) - C] + C$, we have

$$\tilde{W}(u_0) \geq \frac{1}{2} \int_{\Omega} f(\nabla \tilde{u}) dx \quad (2)$$

with equality when $\nabla \tilde{u}$ is in the null space of $\tilde{C}(x) - C$. Suppose we are able to find one solution of the elasticity equations in the medium with tensor C ,

i.e.

$$\nabla \cdot \sigma = 0, \quad \sigma = C \nabla u \quad (3)$$

with $u = u_0$ on $\partial\Omega$: if necessary we could start with a solution to the equations (3) and choose u_0 as the boundary value of u . Then because the quadratic form C is quasiconvex

$$\int_{\Omega} f(\nabla \tilde{u}) dx \geq \int_{\Omega} f(\nabla u) dx = \int_{\partial\Omega} u \cdot (\sigma \cdot n) dS \equiv W(u_0) \quad (4)$$

where n is the outward normal to the surface $\partial\Omega$. To see the condition for equality in (4) define $\delta u = \tilde{u} - u$ inside Ω , then find a cube B containing Ω , set δu to be zero inside the remainder of the cube outside Ω , and finally extend δu to be periodic with this cube B as a unit cell. Then $\nabla \delta u$ has zero average value over the unit cell and since u solves (3),

$$\int_{\Omega} f(\nabla \tilde{u}) - f(\nabla u) dx = \int_B f(\nabla \delta u) dx \quad (5)$$

The condition for this to be zero is easily found using the rank-one convexity and Plancherel's theorem: each Fourier component $\widehat{\delta u}(k)$ of δu must be such that

$$f(\operatorname{Re} \widehat{\delta u}(k) \otimes k) = 0 \quad \text{and} \quad f(\operatorname{Im} \widehat{\delta u}(k) \otimes k) = 0 \quad (6)$$

where $\operatorname{Re} \widehat{\delta u}(k)$ and $\operatorname{Im} \widehat{\delta u}(k)$ are the real and imaginary parts of $\widehat{\delta u}(k)$. Fields $\nabla \delta u$ satisfying this condition are called special fields and a necessary condition for them to exist is that C not be strictly quasiconvex.

In summary we have the inequality

$$\tilde{W}(u_0) \geq W(u_0), \quad (7)$$

which will be sharp when $\nabla \tilde{u}$ is in the null space of $\tilde{C}(x) - C$ and $\nabla \delta u$ is a special field which vanishes in $B \setminus \Omega$. Our chances of finding such fields are greatest when $C_1 - C$ and/or $C_2 - C$ is especially degenerate and when there are lots of special fields which vanish in $B \setminus \Omega$. The last condition is most likely to hold when $f(\xi)$ is extremal, although an example has yet to be produced of an extremal function of gradients, other than a null-Lagrangian, for which there exist special fields which vanish in $B \setminus \Omega$ when Ω is strictly contained in B . However, for bounding the energy stored in a unit cell Ω of a periodic composite with periodic boundary conditions on $\nabla \tilde{u}$ (which is relevant to bounding the effective moduli using the comparison bound) we can take $B = \Omega$ and so any special field automatically vanishes in $B \setminus \Omega$ since $B \setminus \Omega$ is empty.

3 Orthotropic materials

We now briefly introduce orthotropic materials. A homogeneous orthotropic elastic material has three mutually orthogonal planes such that the material properties are symmetric under reflection about each plane. If cartesian coordinate axes are chosen orthogonal to these planes, then the properties are invariant under the transformation $x_a \rightarrow -x_a$, $x_b \rightarrow x_b$, and $x_c \rightarrow x_c$, where abc is permutation of 123. Elements of the elasticity tensor such as C_{abcc} and C_{abbb} in general change sign under such a transformation, so these must be zero. Thus the elements C_{ijkl} of the elasticity tensor must be zero unless the indices $ijkl$ contain an even number of repetitions of the indices 1, 2 or 3. Using the Voigt notation for the elements of C the constitutive law takes the form $\sigma = C\epsilon$ where

$$\sigma = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \quad (8)$$

The mechanical properties are, in general, different along each axis. Orthotropic materials require 9 elastic constants and have as subclasses isotropic materials (with 2 elastic constants), cubic materials (with 3 elastic constants), and transversely isotropic materials (with 5 elastic constants). The wood in a tree trunk is an example of a material which is locally orthotropic: the material properties in three perpendicular directions, axial, radial, and circumferential, are different. Many crystals and rolled metals are also examples of orthotropic materials.

4 Extremal polynomials and relations to the determinants of extremal quasiconvex quadratic forms

In this section we define the notions of *extremality* and *equivalence* of homogeneous polynomials.

Definition 4.1. Assume m and n are natural numbers and $P(x_1, x_2, \dots, x_n)$ is a polynomial of degree $2m$ that is homogeneous of the same degree. Then $P(x)$ is called an extremal polynomial, if $P(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $P(x)$ cannot be written as a sum of two other non-negative polynomials that are linearly independent.

Definition 4.2. Assume m and n are natural numbers and $P(x_1, x_2, \dots, x_n)$ and $Q(x_1, x_2, \dots, x_n)$ are polynomials of degree $2m$ that are homogeneous of the same degree. Then $P(x)$ and $Q(x)$ are equivalent if there exists a non-singular matrix $A \in \mathbb{R}^{n \times n}$ such that $P(x) = Q(Ax)$.

It is then straightforward to prove that this notion of equivalence is actually an equivalence relation preserving also the extremality of polynomials.

Theorem 4.3. The notion of equivalence introduced in Definition 4.2 has the following properties:

- P is equivalent to itself
- If P is equivalent to Q then Q is equivalent to P
- If P is equivalent to Q and Q is equivalent to R , then P is equivalent to R
- If P is equivalent to Q and Q is an extremal then P is an extremal too

As pointed out in introduction our future goal is describing all extremal quasiconvex quadratic forms and the first step to the final goal has been made in [7], where a class of extremals has been found. In order to make progress towards the goal, one asks natural question: What are the properties of extremal quadratic forms? Such a question has not been addressed in [7], but in the present work for the first time. The sought property we believe is the following: The determinant of the y -matrix of the form must be an extremal polynomial, which is not a perfect square. The sufficiency of that statement is proven in the present work for quadratic forms with a linear elastic orthotropic symmetry. Let us now motivate our choice by some examples.

Example 1. The form $f(\xi) = \xi_{11}^2 + \xi_{22}^2 + \xi_{33}^2$ has a determinant of its y -matrix equal to $y_1^2 y_2^2 y_3^2$ which is evidently an extremal polynomial, but $f(\xi)$ is obviously not an extremal.

Example 2. The form $f(\xi) = \xi_{11}^2 + \xi_{22}^2$ has a determinant of its y -matrix equal to 0, which is evidently an extremal polynomial, but $f(\xi)$ is obviously not an extremal.

Example 3. The determinant of the y -matrix of any rank-one form is equivalently zero, but a rank-one form is not an extremal.

Example 4. The most interesting and motivating example, that contains the sought information is the extremal quasiconvex quadratic form

$$Q(\xi) = \xi_{11}^2 + \xi_{22}^2 + \xi_{33}^2 - 2(\xi_{11}\xi_{22} + \xi_{11}\xi_{33} + \xi_{22}\xi_{33}) + \xi_{12}^2 + \xi_{23}^2 + \xi_{31}^2,$$

that appears in [7] (but which does not derive from a tensor having the symmetries of an elasticity tensor).

It turns out that the polynomial

$$P(y) = y_1^4 y_2^2 + y_2^4 y_3^2 + y_3^4 y_1^2 - 3y_1^2 y_2^2 y_3^2$$

that is the determinant of the y -matrix of $Q(\xi)$ is a non-trivial extremal polynomial (by which we mean a polynomial which is not a perfect square). Let us give a proof of that statement.

Proof. Assume in contradiction that the polynomial $P(y)$ is not an extremal. Hence there exists a polynomial $P_1(y)$ such that

$$0 \leq P_1(y) \leq P(y) \quad \text{for all } y \in \mathbb{R}^3, \quad (9)$$

and $P_1(y)$ is not a multiple of $P(y)$. We aim to prove that (9) implies $P_1 = \alpha P$ for some $\alpha \in \mathbb{R}$. It is clear that none of the variables y_i appears in P_1 with power 5 or 6 as otherwise inequality (9) would be violated. The coefficient of y_1^4 in P_1 is a quadratic polynomial in y_2 and y_3 that is less or equal to y_2^2 as inequality (9) implies when $y_1 \rightarrow \infty$, thus it depends only on y_2 , i.e., the coefficient of y_1^4 in P_1 has the form ay_2^2 . Similarly the coefficients of y_2^4 and y_3^4 in P_1 are by_3^2 and cy_1^2 respectively. Thus P_1 has the form

$$\begin{aligned} P_1(y) = & (ay_1^4 y_2^2 + by_2^4 y_3^2 + cy_3^4 y_1^2 + dy_1^2 y_2^2 y_3^2) + a_1 y_1^3 y_2^3 + a_2 y_2^3 y_3^3 + a_3 y_3^3 y_1^3 \\ & + a_4 y_1^3 y_2^2 y_3 + a_5 y_1^3 y_2 y_3^2 + a_6 y_2^3 y_1^2 y_3 + a_7 y_2^3 y_1 y_3^2 + a_8 y_3^3 y_1^2 y_2 + a_9 y_3^3 y_2^2 y_1. \end{aligned}$$

We call the expression in the brackets in P_1 the principal part of P_1 . Note, that changing the sign of any of the variables y_i does not change $P(y)$ but changes

the sign of all summands in P_1 that have an odd power of y_i , thus summing up the inequalities $0 \leq P_1(y_1, y_2, y_3) \leq P(y_1, y_2, y_3)$ and $0 \leq P_1(-y_1, y_2, y_3) \leq P(-y_1, y_2, y_3)$ we get $0 \leq P_2(y) \leq P(y)$ where P_2 has no summands with an odd power of y_1 and has the same principal part as P_1 . Applying the same idea to P_2 for the variable y_2 we end up with the inequality

$$0 \leq ay_1^4y_2^2 + by_2^4y_3^2 + cy_3^4y_1^2 + dy_1^2y_2^2y_3^2 \leq P(y). \quad (10)$$

It is then clear that $0 \leq a, b, c \leq 1$. We have by the Cauchy-Schwartz inequality

$$ay_1^4y_2^2 + by_2^4y_3^2 + cy_3^4y_1^2 \geq 3(abc)^{1/3}y_1^2y_2^2y_3^2$$

and the equality holds for some choice of y , thus we get $3(abc)^{1/3} \geq -d$. On the other hand as $P(1, 1, 1) = 0$, then inequality (10) implies $a + b + c + d = 0$, thus we get

$$3(abc)^{1/3} \geq a + b + c$$

which means that the equality holds in Cauchy-Schwartz, thus $a = b = c$ and $d = -3a$, thus the principal part of $P_1(y)$ is a multiple of $P(y)$. On the other hand testing (9) with $y = (1, t, 0)$ we get

$$at^2 + a_1t^3 \geq 0 \quad \text{for all } t \in \mathbb{R},$$

thus $a_1 = 0$. Similarly we obtain $a_2 = a_3 = 0$. Therefore (9) amounts to the following inequality

$$0 \leq aP(y) + a_4y_1^3y_2^2y_3 + a_5y_1^3y_2y_3^2 + a_6y_2^3y_1^2y_3 + a_7y_2^3y_1y_3^2 + a_8y_3^3y_1^2y_2 + a_9y_3^3y_2^2y_1 \leq P(y) \quad (11)$$

Again, the equalities $P(1, 1, 1) = P(-1, 1, 1)$ and (11) imply $a_6 + a_8 = 0$. Thus if we sum inequalities (11) and the resulting inequality in (11) when changing the sign of y_1 we get

$$|a_6y_1^2y_2y_3(y_2^2 - y_3^2)| \leq \beta P(y) \quad \text{for some } \beta \geq 0.$$

Taking $y_3 = y_1$ in the last inequality we obtain

$$|a_6y_1^3y_2(y_2 - y_1)(y_2 + y_1)| \leq 2\beta y_1^2y_2^2(y_1 - y_2)^2,$$

which implies $a_6 = 0$ if we let $y_1, y_2 \rightarrow 1$ and $y_1 \neq y_2$. As the inequality is symmetric in the variables a_i , then it is straightforward to get $a_i = 0$. Finally we get $P_1(y) = aP(y)$ which is a contradiction. \square

5 Extremal quadratic forms with orthotropic symmetry

The next theorem is the main result of the paper.

Theorem 5.1. *Assume the quadratic form $f(\xi) = (\xi + \xi^T)C(\xi + \xi^T)^T$ depending on the strain has orthotropic symmetry, i.e., the stiffness matrix C has the form (8). Assume furthermore that $C_{11}C_{22}C_{33} \neq 0$. If the determinant of the y -matrix of $f(x, y)$ is an extremal polynomial that is not a perfect square, then f is an extremal form.*

Proof. Assume in contradiction that $f(x, y)$ is not an extremal, then there exists a rank-one form $(x^T B y)^2$ such that

$$f(x, y) - (x^T B y)^2 \geq 0 \quad \text{for all } x, y \in \mathbb{R}^3.$$

Let us now prove that then $f(x, y) = \alpha(x^T B y)^2$ for some $\alpha \geq 1$. Recall the Brunn-Minkowski inequality for determinants [4,5], which will be utilized in the sequel.

Theorem 5.2 (Brunn-Minkowski inequality). *Assume $n \in \mathbb{N}$ and A and B are $n \times n$ symmetric positive semi-definite matrices. Then the following inequality holds:*

$$(\det(A + B))^{1/n} \geq (\det(A))^{1/n} + (\det(B))^{1/n}.$$

Assume now $f(\xi)$ is a quasiconvex quadratic form that has a linear elastic orthotropic symmetry. Then f has the form $f(\xi) = (\xi + \xi^T)^T C (\xi + \xi^T)$, where the stiffness has the form of (8). Thus we get

$$f(\xi) = \sum_{i,j=1}^3 C_{ij} \xi_{ii} \xi_{jj} + C_{44}(\xi_{12} + \xi_{21})^2 + C_{55}(\xi_{13} + \xi_{31})^2 + C_{66}(\xi_{23} + \xi_{32})^2.$$

It is clear that f is then rank-one equivalent to a form

$$F(\xi) = \sum_{i,j=1}^3 a_{ij} \xi_{ii} \xi_{jj} + a_1(\xi_{12}^2 + \xi_{21}^2) + a_2(\xi_{13}^2 + \xi_{31}^2) + a_3(\xi_{23}^2 + \xi_{32}^2),$$

where $a_{ii} = C_{ii}$ and $a_i = C_{jj}$, with $j = i+3$ for $i = 1, 2, 3$. From the inequality

$$F(x, y) - (x^T B y)^2 \geq 0$$

we get that

$$F(x, y) - t(x^T B y)^2 \geq 0 \quad \text{for all } t \in [0, 1]. \quad (12)$$

Denote now by $T(y)$ the y -matrix of the biquadratic form $F(x, y)$ and by $T_t(y)$ the y -matrix of the biquadratic form $F(x, y) - t(x^T B y)^2$. Inequality (12) now implies that the y -matrix of the form $F(x, y) - t(x^T B y)^2$, i.e, the matrix $T_t(y)$ is positive semi-definite for all $y \in \mathbb{R}^3$ and $t \in [0, 1]$. The equality

$$T(y) = [T(y) - T_t(y)] + [T_t(y)],$$

the positive semi-definiteness of the matrices $T(y) - T_t(y)$ and $T_t(y)$ and the Brunn-Minkowski inequality imply

$$(\det(T(y)))^{1/3} \geq (\det(T(y) - T_t(y)))^{1/3} + (\det(T_t(y)))^{1/3},$$

or

$$\det(T(y)) \geq \det(T_t(y)). \quad (13)$$

It is easy to calculate that

$$\det(T_t(y)) = \det(T(y)) - t \sum_{i,j=1}^3 l_i l_j \text{cof}_{ij}(T(y)), \quad (14)$$

where $l_i = \sum_{j=1}^3 b_{ij} y_j$. Inequality (13) now implies

$$\det(T(y)) \geq \det(T(y)) - t \sum_{i,j=1}^3 l_i l_j \text{cof}_{ij}(T(y)).$$

As by the requirement of the theorem $\det(T(y))$ is not identically zero and for $t = 0$ the right hand side of the last inequality is exactly $\det(T(y))$, then by the extremality of $\det(T(y))$ the right hand side must be a multiple of $\det(T(y))$, i.e.,

$$\det(T(y)) - t \sum_{i,j=1}^3 l_i l_j \text{cof}_{ij}(T(y)) = \lambda(t) \det(T(y)),$$

which gives

$$\sum_{i,j=1}^3 l_i l_j \text{cof}_{ij}(T(y)) = \frac{1 - \lambda(t)}{t} \det(T(y)), \quad \text{for } t \neq 0. \quad (15)$$

Both parts of the equality (15) are polynomials in $y = (y_1, y_2, y_3)$ thus the expression $\frac{1-\lambda(t)}{t}$ must be constant, therefore

$$\sum_{i,j=1}^3 l_i l_j \text{cof}_{ij}(T(y)) = d \cdot \det(T(y)), \quad (16)$$

where $d \in \mathbb{R}$. The positive semi-definiteness of $T(y)$ implies positive semi-definiteness of the cofactor matrix $\text{cof}(T(y))$, thus

$$\sum_{i,j=1}^3 l_i l_j \text{cof}_{ij}(T(y)) \geq 0 \quad \text{for all } y \in \mathbb{R}^3.$$

We have on the other hand $\det(T(y)) \geq 0$, and $\det(T(y))$ is not identically zero, thus $d \geq 0$. Consider now two main cases:

Case 1: $d = 0$. In this case identity (16) becomes

$$\sum_{i,j=1}^3 l_i l_j \text{cof}_{ij}(T(y)) \equiv 0.$$

Again, taking into account the positive semi-definiteness of $\text{cof}(T(y))$ we get a system of three identities:

$$l_1 \text{cof}_{i1}(T(y)) + l_2 \text{cof}_{i2}(T(y)) + l_3 \text{cof}_{i3}(T(y)) \equiv 0, \quad i = 1, 2, 3. \quad (17)$$

As the matrix B is different from the zero matrix, it has a rank at least 1, thus the solution to the system of linear equations $l_i = 0, i = 1, 2, 3$ is a proper subspace V of \mathbb{R}^3 , i.e. is included in a hyperplane, which means that the columns of the cofactor matrix $\text{cof}(T(y))$ are linearly dependent in $\mathbb{R}^3 \setminus V$, i.e., $\det(\text{cof}(T(y))) = 0$ for $y \in \mathbb{R}^3 \setminus V$. Therefore, since $\det(\text{cof}(T(y)))$ is continuous in \mathbb{R}^3 it must be zero for all $y \in \mathbb{R}^3$ and by taking the determinant of the identity $T(y)[\text{cof}(T(y))]^T = \det(T(y))I$ we get $\det(T(y)) \equiv 0$, which is a contradiction. **Case 1** is now proved.

Case 2: $d > 0$. In this case identity (16) implies

$$\det(T(y)) = k \sum_{i,j=1}^3 l_i l_j \text{cof}_{ij}(T(y)), \quad k > 0. \quad (18)$$

Our goal is now getting a contradiction from (18). Observe that

$$T(y) = \begin{bmatrix} a_{11}y_1^2 + a_{12}y_2^2 + a_{13}y_3^2 & a_{12}y_1y_2 & a_{13}y_1y_3 \\ a_{12}y_1y_2 & a_{22}y_2^2 + a_{23}y_3^2 + a_{21}y_1^2 & a_{23}y_2y_3 \\ a_{13}y_1y_3 & a_{23}y_2y_3 & a_{33}y_3^2 + a_{21}y_1^2 + a_{32}y_2^2 \end{bmatrix},$$

thus we have the following formulae for the co-factors,

$$\text{cof}_{ii}(T(y)) = P_{ii}(y), \quad (19)$$

and for $\ell \neq m$

$$\text{cof}_{\ell m}(T(y)) = y_\ell y_m P_{\ell m}(y), \quad (20)$$

where P_{ij} is a second or forth degree polynomial in y depending only on y_k^2 . It follows that $\det(T(y))$ is a sixth order polynomial in y depending only on y_i^2 . Recalling the form of the entries of the cofactor matrix and taking into account the fact that the coefficients of the expressions like $y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}$ in the right hand side of (18) are zero, where at least one of the exponents α_i is odd, we get from (18) the following identity:

$$\begin{aligned} \frac{1}{k} \det(T(y)) &= (b_{11}^2 y_1^2 + b_{12}^2 y_2^2 + b_{13}^2 y_3^2) \text{cof}_{11}(T(y)) \\ &\quad + (b_{21}^2 y_1^2 + b_{22}^2 y_2^2 + b_{23}^2 y_3^2) \text{cof}_{22}(T(y)) \\ &\quad + (b_{31}^2 y_1^2 + b_{32}^2 y_2^2 + b_{33}^2 y_3^2) \text{cof}_{33}(T(y)) \\ &\quad + 2(b_{11}b_{22} + b_{12}b_{21})y_1y_2 \text{cof}_{12}(T(y)) \\ &\quad + 2(b_{11}b_{33} + b_{13}b_{31})y_1y_3 \text{cof}_{13}(T(y)) \\ &\quad + 2(b_{22}b_{33} + b_{23}b_{32})y_2y_3 \text{cof}_{23}(T(y)). \end{aligned} \quad (21)$$

The right hand side of identity (21) can be rearranged as follows:

$$\begin{aligned} \frac{1}{k} \det(T(y)) &= (b_{11}y_1, b_{22}y_2, b_{33}y_3) \text{cof}(T(y)) (b_{11}y_1, b_{22}y_2, b_{33}y_3)^T + \\ &\quad + (b_{12}^2 y_2^2 \text{cof}_{11}(T(y)) + 2b_{12}b_{21}y_1y_2 \text{cof}_{12}(T(y)) + b_{21}^2 y_1^2 \text{cof}_{22}(T(y))) \\ &\quad + (b_{13}^2 y_3^2 \text{cof}_{11}(T(y)) + 2b_{13}b_{31}y_1y_3 \text{cof}_{13}(T(y)) + b_{31}^2 y_1^2 \text{cof}_{33}(T(y))) \\ &\quad + (b_{23}^2 y_3^2 \text{cof}_{22}(T(y)) + 2b_{23}b_{32}y_2y_3 \text{cof}_{23}(T(y)) + b_{32}^2 y_2^2 \text{cof}_{33}(T(y))). \end{aligned} \quad (22)$$

Since the cofactor matrix $\text{cof}(T(y))$ is positive semi-definite, then each of the four summands is non-negative. Thus, the extremality of the determinant

$\det(T(y))$, that is a sum of four non-negative polynomials, implies that each summand is either identically zero or a non-zero multiple of the determinant. On the other hand all four summands in (22) cannot be simultaneously identically zero. Consider the following two cases:

Case a: The first summand in (22) is a nonzero multiple of $\det(T(y))$. In this case we have the following representation of the determinant:

$$\begin{aligned} \frac{1}{s} \det(T(y)) &= (b_{11}y_1, b_{22}y_2, b_{33}y_3) \operatorname{cof}(T(y)) (b_{11}y_1, b_{22}y_2, b_{33}y_3)^T \\ &= \sum_{i,j=1}^3 b_{ii}b_{jj}y_iy_j \operatorname{cof}_{ij}(T(y)), \end{aligned} \quad (23)$$

where $s > 0$. Equating the coefficients of y_1^6 , y_2^6 and y_3^6 of the right and left hand sides of (23) we get $a_{ii} = sb_{ii}^2$, for $i = 1, 2, 3$. Now look at $G(\xi) = F(\xi) - s(\sum_{i=1}^3 b_{ii}\xi_{ii})^2$. It has the form

$$G(\xi) = a_1(\xi_{12}^2 + \xi_{21}^2) + a_2(\xi_{13}^2 + \xi_{31}^2) + a_3(\xi_{23}^2 + \xi_{32}^2) + 2b_1\xi_{11}\xi_{22} + 2b_2\xi_{11}\xi_{33} + 2b_3\xi_{22}\xi_{33}.$$

and the determinant of its y -matrix is

$$\begin{aligned} \det(T_G(y)) &= (a_1^2a_2 - b_1^2a_2)y_1^4y_2^2 + (a_1^2a_3 - b_1^2a_3)y_1^2y_2^4 \\ &\quad + (a_2^2a_1 - b_2^2a_1)y_1^4y_3^2 + (a_2^2a_3 - b_2^2a_3)y_1^2y_3^4 \\ &\quad + (a_3^2a_1 - b_3^2a_1)y_2^4y_3^2 + (a_3^2a_2 - b_3^2a_2)y_2^2y_3^4 \\ &\quad + 2(a_1a_2a_3 + b_1b_2b_3)y_1^2y_2^2y_3^2. \end{aligned} \quad (24)$$

On the other hand by analogy with the formula (14) (with $t = s$ and B being diagonal) we have

$$\det(T_G(y)) = \det(T(y)) - s \sum_{i,j=1}^3 b_{ii}b_{jj}y_iy_j \operatorname{cof}_{ij}(T(y)) = 0.$$

Consider now the three different subcases:

Case a1: For all $i = 1, 2, 3$ there holds $a_i > 0$. It is clear that $\det(T_G(y))$ is identically zero if and only if $|b_i| = a_i$ and $a_1a_2a_3 = -b_1b_2b_3$. Therefore there are two cases possible: Either all b_i are negative, or two of

them are positive and the other one is negative. Note, that we can change the sign of any x_i that will change the signs of two of b_i , which means that one can without loss of generality assume that all b_i are negative. Next, the substitution $x_i = \lambda x'_i$ and $y_i = \lambda y'_i$, where $\lambda_i \neq 0$ changes each a_i by the factor λ_i^2 , transferring G to a rank-one equivalent quadratic form of the same form with $a_i = 1$. The above non-singular transformations do not change the form of a linear combination of the variables ξ_{11} , ξ_{22} and ξ_{33} , thus we end up with the formula for $F(\xi)$ up to rank-one equivalence:

$$F(\xi) = (a\xi_{11} + b\xi_{22} + c\xi_{33})^2 + (\xi_{12} - \xi_{21})^2 + (\xi_{13} - \xi_{31})^2 + (\xi_{23} - \xi_{32})^2. \quad (25)$$

It is straightforward to calculate

$$\det(T_F(y)) = y_1^2(ay_1^2 + by_2^2 + cy_3^2)^2 + y_2^2(ay_1^2 + by_2^2 + cy_3^2)^2 + y_3^2(ay_1^2 + by_2^2 + cy_3^2)^2,$$

which is not an extremal polynomial unless $ay_1^2 + by_2^2 + cy_3^2$ is identically zero, i.e., $a = b = c = 0$, thus we get $C_{11} = 0$ which is a contradiction. **Case a1** is proved.

Case a2: $a_3 = 0$, $a_1, a_2 > 0$. In this case we again obtain from $\det(T_G(y)) \equiv 0$, that $|b_i| = a_i$, for $i = 1, 2, 3$. The same argument as in the previous case leads to a situation

$$F(\xi) = (a\xi_{11} + b\xi_{22} + c\xi_{33})^2 + (\xi_{12} + \xi_{21})^2 + (\xi_{13} + \xi_{31})^2. \quad (26)$$

Observe, that from the form of $F(\xi)$ we have

$$\det(T_F(y)) = y_1^2 P(y),$$

where $P(y)$ is a fourth degree polynomial of y . Hilbert's theorem [8] asserts that any fourth degree non-negative homogeneous polynomial in three variables is a sum of squares of degree two polynomials. Next, we have that $P(y) \geq 0$, thus as $\deg(P) = 4$, then by Hilbert's theorem $P(y)$ is a sum of squares of second degree polynomials, which means that $\det(T_F(y)) = y_1^2 P(y)$ is either a perfect square or not an extremal, which is a contradiction. **Case a2** is proved.

Case a3: $a_2 = a_3 = 0$, $a_1 > 0$. In this case $\det(T_G(y)) \equiv 0$ implies $b_2 = b_3 = 0$ thus we arrive at

$$F(\xi) = (a\xi_{11} + b\xi_{22} + c\xi_{33})^2 + \xi_{12}^2 + \xi_{21}^2 + 2d\xi_{12}\xi_{21}. \quad (27)$$

Like in the previous case, we again have $\det(T_F(y)) = y_1^2 P(y)$, and the same argument as in **Case a2**, completes the proof.

Case a3: $a_1 = a_2 = a_3 = 0$. In this case we have

$$F(\xi) = \sum_{i,j=1}^3 a_{ij} \xi_{ii} \xi_{jj},$$

thus

$$\det(T_G(y)) = ay_1^2 y_2^2 y_3^2,$$

which is again a contradiction. **Case a** is now completely proved.

Case b: The second summand in (22) is a nonzero multiple of $\det(T(y))$. Observe, that we get in this case

$$\frac{1}{s} \det(T(y)) = (b_{21}y_1, b_{12}y_2, 0 \cdot y_3) \operatorname{cof}(T(y)) (b_{21}y_1, b_{12}y_2, 0 \cdot y_3)^T \quad (28)$$

for some $s > 0$, therefore **Case b** reduces to **Case a**. The theorem is proved now. \square

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References

- [1] G. Allaire and R.V. Kohn. Optimal lower bounds on the elastic energy of a composite made from two non-well-ordered isotropic materials, *Quarterly of applied mathematics*, vol. LII, 2 (1994), 331–333.
- [2] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.*, 63 (1977), 337–403.

- [3] J. M. Ball. Remarks on the paper: "Basic calculus of variations", *Pacific Journal of Mathematics*, 116 (1985), 7–10.
- [4] E. F. Beckenbach and R. Bellman, *Inequalities*, Springer, Berlin, 1961.
<http://dx.doi.org/10.1007/978-3-642-64971-4>
- [5] R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York, 1960.
- [6] B. Dacorogna. *Direct methods in the calculus of variations* 2nd Edition, 2008 Springer Science+Business Media, LLC.
- [7] D. Harutyunyan and G.W. Milton. Examples of extremal quasiconvex quadratic forms that are not polyconvex, *submitted*.
- [8] D. Hilbert. Über die Darstellung definiter Formen als Summen von Formenquadraten. *Math. Ann.* 32 (1888), 342–350.
- [9] H. Kang and G. W. Milton. Bounds on the volume fractions of two materials in a three dimensional body from boundary measurements by the translation method, *SIAM Journal on Applied Mathematics*, 73 (2013), 475–492.
- [10] K. A. Lurie and A. V. Cherkhev. Accurate estimates of the conductivity of mixtures formed of two materials in a given proportion (two-dimensional problem), *Doklady Akademii Nauk SSSR*, 264 (1982), 1128–1130, English translation in *Soviet Phys. Dokl.* 27 (1982), 461–462.
- [11] K. A. Lurie and A. V. Cherkhev. Exact estimates of conductivity of composites formed by two isotropically conducting media taken in prescribed proportion *Proceedings of the Royal Society of Edinburgh, Section A, Mathematical and Physical Sciences* 99 (1984), 71–87.
- [12] G. W. Milton. On characterizing the set of positive effective tensors of composites: The variational method and the translation method, *Communications on Pure and Applied Mathematics*, Vol. XLIII (1990) 63–125.
- [13] G. W. Milton. *The Theory of Composites* vol. 6 of Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, United Kingdom, 2002

- [14] G.W. Milton. Sharp inequalities which generalize the divergence theorem—an extension of the notion of quasiconvexity , *Proceedings Royal Society A* 469 (2013), 20130075: see also the Addendum in arXiv:1302.0942v4 [math.AP].
- [15] C. B. Morrey. Quasiconvexity and the lower semicontinuity of multiple integrals, *Pacific Journal of Mathematics* 2 (1952) 25–53
- [16] C. B. Morrey. *Multiple integrals in the calculus of variations*, Springer–Verlag, Berlin, 1966.
- [17] F. Murat and L. Tartar. Calcul des variations et homogénéisation. (French) [Calculus of variation and homogenization], in Les méthodes de l’homogénéisation: théorie et applications en physique, volume 57 of Collection de la Direction des études et recherches d’Electricité de France, pages 319–369, Paris, 1985, Eyrolles, English translation in Topics in the Mathematical Modelling of Composite Materials, pp. 139–173, ed. by A. Cherkaev and R. Kohn, ISBN 0-8176-3662-5.
- [18] L. Tartar, Estimation de coefficients homogénéisés (French) [Estimation of homogenization coefficients], in Computing Methods in Applied Sciences and Engineering: Third International Symposium, Versailles, France, December 5–9, 1977, pages 364–373, Springer-Verlag, Berlin, 1979, English translation in Topics in the Mathematical Modelling of Composite Materials, pp. 9–20, ed. by A. Cherkaev and R. Kohn, ISBN 0-8176-3662-5.
- [19] L. Tartar, Estimations fines des coefficients homogénéisés. (French) [Fine estimations of homogenized coefficients], in Ennio de Giorgi Colloquium: Papers Presented at a Colloquium held at the H. Poincaré Institute in November 1983, edited by P. Krée, volume 125 of Pitman Research Notes in Mathematics, pages 168–187, London, 1985, Pitman Publishing Ltd.
- [20] F. J. Terpstra. Die Darstellung biquadratischer Formen als Summen von Quadraten mit Anwendung auf die Variationsrechnung, *Mathematische Annalen* 116 (1938), 166–180.
- [21] D. Serre. Condition de Legendre-Hadamard: Espaces de matrices de rang $\neq 1$. (French) [Legendre-Hadamard condition: Space of matrices

of rank $\neq 1$], Comptes rendus de l'Académie des sciences, Paris 293 (1981), 23-26.

- [22] D. Serre. Formes quadratiques et calcul des variations (French) [Quadratic forms and the calculus of variations], *Journal de Mathématiques Pures et Appliquées* 62 (1983), 177-196.
- [23] L. Van Hove. Sur l'extension de la condition de Legendre du calcul des variations aux intégrales multiples à plusieurs fonctions inconnues, *Nederl. Akad. Wetensch. Proc.* 50 (1947), 18-23.
- [24] L. Van Hove. Sur le signe de la variation seconde des intégrales multiples à plusieurs fonctions inconnues, *Acad. Roy. Belgique Cl. Sci. Mém. Coll.* 24 (1949), 68.